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Author(s): C. F. Menezes and D. L. Hanson

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ON THE THEORY OF RISK AVERSION*

BY C. F. MENEZES AND D. L. HANSON[1]

1. INTRODUCTION

AN IMPORTANT ADVANCE in the economics of uncertainty has been the development of the theory of risk aversion, due to independent seminal contributions by K. J. Arrow [1], [2] and by J. W. Pratt [4]. This paper extends the existing theory by establishing the economic significance of the partial relative risk aversion function. Let $u(t)$ be a utility function for wealth. The functions $A(t) = -u''(t)/u'(t)$ and $R(t) = -tu''(t)/u'(t)$ are the Arrow-Pratt absolute and relative risk aversion functions. The importance of $A(t)$ arises when considering an individual's aversion to risk as wealth is varied but the risk remains unchanged, while $R(t)$ becomes relevant when wealth and the risk are changed in the same proportion. We shall demonstrate that the partial relative risk aversion function $P(t; w) = -tu''(t + w)/u'(t + w)$ is important when the risk is varied but wealth w remains fixed. In addition, we indicate the economic relationships between the functions A, R, and P; present some results about the behavior of P; and relate its behavior to that of A and R.

The analysis in this paper is based on Pratt's risk premium which we feel is the only function which actually measures risk aversion for arbitrary risks.[2] Our analysis differs from that of both Arrow and Pratt in that we do not use "infinitesimal" risks. Arrow and Pratt interpret A and R as "local" measures of absolute and relative risk aversion. The results of this paper show that the functions A, R, and P have a significance beyond their interpretation as "local" measures of risk aversion in that they determine the behavior of the risk premium in different comparative static contexts.

In Section 2 basic concepts are briefly discussed. Section 3 contains our main results. In it the economic significance of A, R, and the new function P is established through their relationship with the risk premium, and we show how the behavior of these functions is relevant for the theory of risk aversion. Section 4 contains a comparison of our results with those of Arrow and Pratt, and indicates the usefulness of A, P, and R for comparative static analysis of expected utility maximization models.

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[2] Arrow's [2] "more-than-fair odds" concept measures risk aversion for binary risks $z = \pm h$; that is, risks where one bets to gain or lose a fixed amount $h > 0$.

2. BASIC CONCEPTS

We begin by defining risk aversion. An individual is a risk averter if for any arbitrary risk he prefers the sure amount equal to the expected value of the risk to the risk itself. Let w be his initial wealth and z, a random variable, be his risky prospect. He is risk averse if

$$u[w + E(z)] > E[u(w + z)] \tag{2.1}$$

where E is the expectation operator. A necessary and sufficient condition for (2.1) to hold for all values of w and all risks z is that u be strictly concave, or equivalently that it be the integral of a strictly decreasing marginal utility of wealth function u'.

A general measure of risk aversion is Pratt's risk premium $\pi(w, z)$ defined by the equation

$$u[w + E(z) - \pi(w, z)] = E[u(w + z)] . \tag{2.2}$$

As the notation suggests, the risk premium $\pi(w,z)$ depends both on w and on the distribution of z. $\pi(w, z)$ can be interpreted as the maximum amount, beyond the negative of the expected value of the risk itself, which an individual with wealth equal to w would pay to insure against the risk z. In view of (2.1), $\pi(w, z)$ is positive for a risk averter. Since u is monotonically increasing, its inverse u^{-1} exists and (2.2) can be solved for $\pi(w, z)$ giving

$$\pi(w, z) = w + E(z) - u^{-1}[Eu(w + z)] . \tag{2.3}$$

In this form it can be seen that the risk premium is the amount by which the expected value of the risk $(w + z)$ exceeds its cash equivalent $u^{-1}[Eu(w + z)]$. This interpretation and that given above are alternative ways of stating that the risk premium $\pi(w, z)$ is the amount needed to make the individual indifferent between the risk z and the sure amount $E(z) - \pi(w, z)$. From either (2.2) or (2.3) it can be seen that the risk premium is invariant under positive linear transformations of the utility function. This property of $\pi(w, z)$ indicates that our interpretation of the risk premium as a general measure of risk aversion is consistent with von Neumann-Morgenstern utility theory.

The significance of the functions A, P, and R arises when considering the the variation in $\pi(w, z)$ with respect to w and the risk z. The effects on the risk premium of a change in wealth can be determined from the sign of $(\partial/\partial w)\pi(w, z)$. In the next section we show that the sign of this partial derivative depends on the behavior of the absolute risk aversion function A.

The notion of variation in the risk z requires careful specification since it is the distribution of z that is being varied. Two simple types of variation can be studied, the first being an additive shift accomplished by replacing the risk z by the risk $z + h$ where h is constant. From (2.3) it can be seen that

$$\pi(w + h, z) = \pi(w, z + h) . \tag{2.4}$$

Hence, the effect on the risk premium of an additive shift in the distribution of z depends on the behavior of the absolute risk aversion function A.

The second type of variation involves replacing the risk z by the risk λz in (2.3) and elsewhere. It should be noted that, unless z is actuarially neutral ($E(z) = 0$), this multiplicative transformation gives a change in dispersion accompanied by an implicit change in wealth. From (2.4) it follows that

$$\pi(w, \lambda z) = \pi[w + (\lambda - 1)E(z), \lambda z + (1 - \lambda)E(z)] , \tag{2.5}$$

where $(\lambda - 1)E(z)$ is the implicit increase in wealth and $z_1 = \lambda z + (1 - \lambda)E(z)$ is a risk with the same expected value as z but with greater dispersion.

In the next section we study the effect on the risk premium of a multiplicative increase in z and show that the behavior of the partial relative risk aversion function P determines the sign of $(\partial/\partial\lambda)[\pi(w, \lambda z)/\lambda]$. This sign has a straightforward economic meaning: it provides information about the elasticity of the risk premium with respect to the multiplicative factor used to transform the risk. This interpretation follows from the fact that $(\partial/\partial\lambda)[\pi(w, \lambda z)/\lambda] \gtreqless 0$ implies $\partial \log \pi(w, \lambda z)/\partial \log \lambda \gtreqless 1$.

Finally, the effect on the risk premium when wealth and the risk are varied in the same proportion can be studied. We will show that the behavior of the relative risk aversion function R determines the sign of $(\partial/\partial\lambda)[\pi(\lambda w, \lambda z)/\lambda]$. This sign provides information about the elasticity of the risk premium $\pi(\lambda w, \lambda z)$ with respect to λ, the factor by which *both* wealth and the risk are multiplied. This interpretation follows from the fact that $(\partial/\partial\lambda)[\pi(\lambda w, \lambda z)/\lambda] \gtreqless 0$ implies $\partial \log \pi(\lambda w, \lambda z)/\partial \log \lambda \gtreqless 1$. In contrast, recall that the sign of $(\partial/\partial\lambda)[\pi(w, \lambda z)/\lambda]$ provides information about what might appropriately be called the partial elasticity of the risk premium since it indicates the proportionate change in the risk premium resulting from a proportionate change in the multiplicative factor used to transform the risk, wealth being held constant.

3. THE MAIN RESULTS

In this section the economic significance of the risk aversion functions A, P, and R is established through their relationship with the risk premium $\pi(w, z)$. The theorem presented below gives our main result and shows why the behavior of these functions is important for the theory of risk aversion. Later in this section we present some results about the behavior of the partial relative risk aversion function P and relate its behavior to that of the absolute risk aversion function A and to that of the relative risk aversion function R.

The following lemma is used to prove the theorem.

LEMMA.[3] *Let u be a utility function with inverse function u^{-1} and derivatives u' and u''. The functions*

(i) $$\psi_A(t) = u'[u^{-1}(t)] ,$$

(ii) $$\psi_P(t) = u^{-1}(t)u'[u^{-1}(t)] - wu'[u^{-1}(t)] ,$$

[3] Convex, concave, increasing, and decreasing are used in the strict sense here and in what follows. Similar results can be obtained when they are not used in the strict sense.

(iii) $$\psi_R(t) = u^{-1}(t)u'[u^{-1}(t)]\,,$$

are concave, linear, or convex in t depending on whether the corresponding functions (i) $A(t) = -u''(t)/u'(t)$, (ii) $P(t; w) = -tu''(t+w)/u'(t+w)$, (iii) $R(t) = -tu''(t)/u'(t)$, *are increasing, constant, or decreasing in t.*

PROOF. The proof follows immediately by differentiation of the functions ψ_A, ψ_P, and ψ_R. By way of example we show the validity of (iii). Using the definition of ψ_R we get

$$\psi'_R(t) = 1 + u^{-1}(t)\frac{u''[u^{-1}(t)]}{u'[u^{-1}(t)]}\,. \tag{3.1}$$

Now since u is strictly increasing, so is u^{-1}. Thus $\psi'_R(t)$ is increasing, constant, or decreasing (as a function of t) if and only if $su''(s)/u'(s)$ is increasing, constant, or decreasing respectively (as a function of s). By definition $R(s) = -su''(s)/u'(s)$. Hence, it follows that ψ_R is concave, linear, or convex if and only if R is increasing, constant, or decreasing, respectively.

In the theorem that follows we assume z is a random variable with distribution F defined on an interval $[a, b]$. For each w we restrict a and the multiplicative factor λ so that $Pr\{w + \lambda z < 0\} = 0$. Economically, this means that the individual is restricted to risks which exclude the possibility of losing more than his initial wealth.

THEOREM. *Let* $\pi(w, z)$ *be defined as in* (2.3). *Then*

(i) $$\frac{\partial}{\partial w}\pi(w, z) \gtreqless 0\,,$$

(ii) $$\frac{\partial}{\partial \lambda}\left[\frac{\pi(w, \lambda z)}{\lambda}\right] \gtreqless 0\,,$$

(iii) $$\frac{\partial}{\partial \lambda}\left[\frac{\pi(\lambda w, \lambda z)}{\lambda}\right] \gtreqless 0\,,$$

if the corresponding function (i) *absolute risk aversion* $A(t)$, (ii) *partial relative risk aversion* $P(t; w)$, (iii) *relative risk aversion* $R(t)$, *is respectively increasing, constant, or decreasing in t.*

PROOF. We show the validity of part (ii) of the theorem. Proofs for parts (i) and (iii) use similar arguments and are therefore omitted. Using (2.3) and differentiating with respect to λ gives

$$\frac{\partial}{\partial \lambda}\left[\frac{\pi(w, \lambda z)}{\lambda}\right] \gtreqless 0 \tag{3.2}$$

if and only if

$$u'[u^{-1}(Eu(w + \lambda z))][u^{-1}(Eu(w + \lambda z)) - w] \gtreqless E[u'(w + \lambda z)\lambda z]\,. \tag{3.3}$$

Now consider the function $\psi_P(t) = u^{-1}(t)u'[u^{-1}(t)] - wu'[u^{-1}(t)]$. By Jensen's inequality $\psi_P[E(t)] \gtreqless E[\psi_P(t)]$ if ψ_P is respectively concave, linear, or convex in t. Let $t = u(w + \lambda z)$. Then (3.3) reduces to $\psi_P(Et) \gtreqless E[\psi_P(t)]$. Hence $(\partial/\partial\lambda)[\pi(w, \lambda z)/\lambda] \gtreqless 0$ according to whether the function $\psi_P(t)$ is concave, linear, or convex. But by part (ii) of the lemma, $\phi_P(t)$ is concave, linear, or convex if and only if $P(t; w)$ is respectively increasing, constant, or decreasing in t. This proves the theorem.

The theorem shows the role of the functions A, R, and P in the theory of risk aversion. The behavior of the absolute risk aversion A gives information about the behavior of the risk premium when wealth is varied but the risk is fixed; the behavior of the relative risk aversion R gives information about the behavior of the proportional change in the risk premium when wealth and the risk are changed in the same proportion; and finally, the behavior of partial relative risk aversion P gives information about the behavior of the proportional change in the risk premium resulting from a given proportional change in the risk, wealth remaining fixed.

In the remainder of this section we briefly discuss the behavior of the functions A, R, and P under the usual assumption that u belongs to the class of monotone increasing concave utility functions. The absolute risk aversion A may increase, decrease, remain constant, and may even be non-monotone. Arrow [2] has formulated the hypothesis of decreasing absolute risk aversion. The hypothesis seems plausible since decreasing A means that one will buy less insurance against a given risk as wealth increases.

Similarly, the relative risk aversion R may increase, decrease, remain constant, or be non-monotone. Arrow has formulated the hypothesis of increasing relative risk aversion. It follows from the theorem that increasing R means that the elasticity of the risk premium with respect to the multiplicative factor by which both wealth and the risk are increased is greater than unity, or equivalently that the proportion of wealth spent for insurance increases when wealth and the risk are increased in the same proportion.

We now present some results about the behavior of the partial relative risk aversion $P(t; w)$. Proofs are omitted since they can be found in [3].

1) *Fix* w. *If* $P(t; w)$ *is non-increasing in* t *for* t *is some interval* $(0, t_0)$ *with* $t_0 > 0$, *then either* $P(t; w) = 0$ (*and consequently* $u''(t + w) = 0$ *for* $0 < t < t_0$) *or else* $w = 0$.

2) *Fix* $w > 0$ *and suppose* $t_0 > 0$. *If* $P(t; w)$ is *monotone* (*strictly monotone*) *in* t *for* $0 < t < t_0$, *then it is non-decreasing* (*strictly increasing*) *there.*

3) *If* $R(t)$ *is non-decreasing then either* $u''(t) = 0$ (*so that* u *is linear*), *or else* $P(t; w)$ *is a strictly increasing function of* t *for each given* w.

The economic meaning of these propositions is straightforward. The first two indicate that if the individual has some positive initial wealth and the partial relative risk aversion is monotone, then either $P(t; w)$ is strictly increasing in t or else the individual is neutral to risk.[4] The assumption of risk aversion rules out the latter possibility. Unfortunately, there remains the possibility that $P(t; w)$ may be non-monotone. However, if Arrow's hypothesis of increasing relative risk aversion is accepted, then it follows from Proposition 3 that partial relative risk aversion is strictly increasing. Thus either $P(t; w)$ monotone or $R(t)$ increasing guarantees that $P(t, w)$ is strictly

[4] More precisely, there is a positive $\dot{t}$ and an actuarially neutral risk z such that the individual is indifferent between the sure amount t and the risk $(z + t)$.

increasing in t. Theoretical considerations therefore suggest the hypothesis of increasing P. Moreover, the hypothesis seems to be intuitively plausible since it amounts to assuming that a given proportional increase in the risk, wealth remaining fixed, will result in a more than proportional increase in the risk premium.

4. SUMMARY AND COMPARISON WITH THE LITERATURE

The importance of the functions A and R for the theory of risk aversion was discovered independently by K. J. Arrow [1], [2], and by J. W. Pratt [4]. We now briefly discuss their contributions.

Using different methods Pratt gets results that are similar to those presented in parts (i) and (iii) of our theorem. However, his derivation does not yield our interpretation that the behavior of R is important because it determines the elasticity of the risk premium with respect to the multiplicative factor used to vary both wealth and the risk. To interpret A and R, Pratt shows that under suitable regularity conditions on the utility function and for "small" risks, A and R can be interpreted as "local" measures of absolute and proportional (relative) risk aversion. Specifically

$$\pi(w, z) \approx (\sigma^2/2)\, A(w + E(z)) \,.$$

To interpret R, Pratt measures the risk premium and the risk itself not in absolute terms but as proportions of wealth. Let $\pi^* = \pi/w$ and $z^* = z/w$ denote the proportional risk premium and the proportional risk, respectively. Then Pratt shows that

$$\pi^*(w, z^*) \approx (\sigma^2/2) R(w + wE(z^*))$$

where σ^2 is now the variance of z^*.

Arrow's analysis deals with binary prospects $z = \pm h$; that is, where the risk is a bet to gain or lose a fixed amount $h > 0$. To show the significance of A for the theory of risk aversion, Arrow [2] considers the probability p^* such that an individual is indifferent between the status quo and winning or losing a bet with probabilities p^* and $1 - p^*$, respectively. Under suitable regularity conditions on the utility function and for "small" bets h, Arrow shows that

$$p^* \approx \frac{1}{2} + \frac{h}{4} A(w) \,.$$

To establish the significance of R, Arrow measures the bet not in absolute terms but as a proportion of wealth. Let $h^* = h/w$, so that h^* is the fraction of wealth at stake. Arrow shows that

$$p^* \approx \frac{1}{2} + \frac{h^*}{4} R(w) \,.$$

This paper has shown that the partial relative risk aversion function P is important for the theory of risk aversion. In addition, we demonstrated that the functions A and R have a significance beyond their interpretation as "local" measures of risk aversion. Our formulation shows that a major role of the functions A, P, and R in the theory of risk aversion is in establishing the behavior of the risk premium in different comparative static contexts.

As might be expected, the functions A, P, and R are also important for comparative static analysis of expected utility maximization models. For example, Arrow [2] shows that in a two-asset portfolio model decreasing A implies that the *amount* of risky investment increases with wealth, while increasing R implies that the *proportion* of wealth in the risky investment decreases with increasing wealth. Similarly, it is shown in [3] that in an expected utility maximization bidding model, the behavior of A determines the direction of change in the optimal bid price with respect to a change in initial wealth, while the behavior of P determines the direction of change in the optimal bid price with respect to changes in contract size.

University of Missouri, Columbia, U.S.A.

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